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# Computation of Multiply Connected Schwarz-Christoffel Maps for Exterior Domains

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**Abstract.** We have recently derived a Schwarz-Christoffel formula for the conformal mapping of the exterior of a finite number of disks to the exterior of a set of polygonal curves [5]. In this work we show how to formulate a set of equations for determining the parameters of such a map. A number of examples are computed, including exteriors of multiple slits. We also recall the derivation of the mapping formulae and give a new formula for the doubly connected case.

**Keywords.** Schwarz-Christoffel transformation, conformal mapping.

**2000 MSC.** Primary 30C30; Secondary 65E05.

## 1. Introduction

Three of the authors have shown that the Schwarz-Christoffel mapping formula can be generalized from the simply connected setting to the multiply connected setting for any finite connectivity [5]. In this work we show how to formulate a finite set of equations to determine the unknown parameters in a conformal map from the exterior of $m$ circles to the exterior of $m$ polygonal curves. (As in the case of the usual Schwarz-Christoffel map, these can include doubly covered slits, which are important for some applications.) We describe a numerical implementation of the solution of these equations and present examples which show our progress in creating a general numerical algorithm for multiply connected Schwarz-Christoffel maps.

This paper will consider only exterior maps, but a similar, and simpler, formula has been derived for bounded domains, and this will be reported by the first author in a separate work [3]. There, as here, the key idea in the derivation is the use of reflection to obtain the correct singularity function $S(z)$ which is shown to be equal to $f''(z)/f'(z)$. In this regard, we mention the recent work

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of Crowdy [2] in which an alternative formula is derived for bounded domains. In that work special functions, called Schottky-Klein prime functions, are introduced into the problem, and a formula which generalizes the known formula for doubly connected domains in terms of theta functions is given. The connection between that formula and the one obtained by reflection will be clarified in the above mentioned work of the first author [3].

The remainder of the paper is organized as follows: In Section 2, we review the derivation of the mapping formulae for the exterior simply, doubly, and multiply connected cases using our reflection argument. We include a new formula for the doubly connected case. In Section 3, we formulate a non-linear set of equations for computing the parameters of the mapping function. In Section 4, we discuss other aspects of our numerical algorithm and give several computational examples.

## 2. The reflection argument

**2.1. Preliminary review of simply and doubly connected problems.** Suppose that $f$ is a conformal map of $\Omega = \{|z| > 1\}$ onto the exterior of a $K$-sided bounded polygon in the $w$-plane with vertices $w_k, k = 1, \ldots, K$ and $f(\infty) = \infty$. Denote the $w$-plane region by $P$ and the prevertices on $|z| = 1$ by $z_k$ with $f(z_k) = w_k$. The angle of rotation from side $\overrightarrow{w_{k-1}w_k}$ to $\overrightarrow{w_k w_{k+1}}$ in the counterclockwise direction is $\alpha_k \pi$. We define the turning parameters $\beta_k = \alpha_k - 1$, then $\beta_k \pi = \arg(w_{k+1} - w_k) - \arg(w_k - w_{k-1})$ is the turning of the tangent at the $k$-th vertex of the polygon. The closure of the polygon requires

$$\sum_{k=1}^{K} \beta_k = 2.$$

Then we have to show that

$$\frac{f''(z)}{f'(z)} = -\frac{2}{z} + \sum_{k=1}^{K} \frac{\beta_k}{z - z_k}, \qquad z \in \Omega.$$

The exterior Schwarz-Christoffel formula

$$f'(z) = c \prod_{k=1}^{K} \left(1 - \frac{z_k}{z}\right)^{\beta_k}$$

follows directly from this by integrating and exponentiating.

The key then is to show that the pre-Schwarzian $f''(z)/f'(z)$ has the form shown. When $f(z)$ is continued analytically by reflection (an even number of times) across an arc of $|z| = 1$ between prevertices, a function $\tilde{f}(z) = a\overline{f(1/\overline{z})} + b$ is obtained for some constants $a$ and $b$. This transformation leaves the pre-Schwarzian invariant. Then $f''(z)/f'(z)$ is analytic on $C \setminus \{0, z_1, \ldots, z_n\}$. The point 0 enters the picture here because $\infty$ is an interior point of $\Omega$ and $f$ has a simple pole there.

The usual local argument, e.g. [9, p. 478], shows $\tilde{f}''(z)/\tilde{f}'(z) - \sum_{k=1}^{m} \beta_k/(z - z_k)$ is analytic in $\mathbb{C} \setminus \{0\}$. If we write

$$f(z) = cz + c_0 + \frac{c_1}{z} + \frac{c_2}{z^2} + \cdots, \qquad c \neq 0,$$

we can say, after reflection through $|z| = 1$,

$$\frac{\tilde{f}''(z)}{\tilde{f}'(z)} = -\frac{2}{z} - \frac{2\overline{c}_1}{\overline{c}} z + \cdots$$

for $z$ near 0, so that

$$\frac{\tilde{f}''(z)}{\tilde{f}'(z)} + \frac{2}{z} - \sum_{k=1}^{K} \frac{\beta_k}{z - z_k} =: E(z)$$

is an entire function. The familiar argument makes the observation at this point that $E(z)$ is analytic at $\infty$ with $E(\infty) = 0$ and applies Liouville's Theorem. It is useful for us to point out an alternative argument. We use the following steps.

1) If $w(t) = f(e^{it})$, the tangent angle on the image polygon is

$$\psi(t) = \arg(w'(t)) = \arg(ie^{it} f'(e^{it}))$$

and since $\psi$ is piecewise constant

$$\psi'(t) = 1 + \operatorname{Re}\left( e^{it} \frac{f''(e^{it})}{f'(e^{it})} \right) = 1 + \operatorname{Re}\left( z \frac{f''(z)}{f'(z)} \right) = 0$$

between prevertices.

2)

$$\operatorname{Re}\left( -z \sum_{k=1}^{K} \frac{\beta_k}{z - z_k} \right) = -\sum_{k=1}^{K} \beta_k \operatorname{Re}\left( \frac{z\overline{z}_k}{z\overline{z}_k - 1} \right) = -\sum_{k=1}^{K} \beta_k \frac{1}{2} = -1$$

on $|z| = 1, z \neq z_k$ since $\operatorname{Re}(z/(z-1)) = 1/2$ for $|z| = 1$.

3) From 1) and 2) $\operatorname{Re}(zE(z)) = 0$ for $z \neq z_k$, and then, by continuity, for $|z| = 1$.

4) The harmonic function $\operatorname{Re}(zE(z))$ is zero on $|z| = 1$. For large $|z|$ we can write

$$z \frac{f''(z)}{f'(z)} = \frac{2\frac{c_1}{c}}{z^2} + \frac{6\frac{c_2}{c}}{z^3} + \cdots$$

and

$$z \left( \frac{2}{z} - \sum_{k=1}^{K} \frac{\beta_k}{z - z_k} \right) = -\frac{1}{z} \sum_{k=1}^{K} \frac{\beta_k z_k}{1 - \frac{z_k}{z}},$$

so that $\operatorname{Re}(zE(z))$ is zero on $\Omega$, and $zE(z)$ is constant there. The constant is zero by virtue of the above identities.

We note for future reference that, since the expansions at $\infty$ of $f''(z)/f'(z)$ and $E(z)$ must agree, we have

$$\frac{2\frac{c_1}{c}}{z^3}+\frac{6\frac{c_2}{c}}{z^4}+\cdots=-\frac{2}{z}+\sum_{k=1}^{K}\frac{\beta_k}{z-z_k}=\left(\sum_{k=1}^{K}\beta_k z_k\right)z^{-2}+\cdots$$

so that the constraint

$$\sum_{k=1}^{K}\beta_k z_k=0 \tag{1}$$

necessarily holds. There is some question about whether this constraint should be included as an equation in a numerical algorithm or if it should be used as a check on computations. We will return to this question in a more general context later.

It is also useful for us to generalize this formula to the case when $f(p)=\infty$ where $p$ is a finite point in $\Omega$. Since $f$ has a simple pole at $p$ we can deduce that

$$\frac{f''(z)}{f'(z)}\sim-\frac{2}{z-p}$$

for $z$ near $p$ and when $f$ is reflected across an arc of $|z|=1$, $p$ is reflected to $p^*=1/\overline{p}$, and

$$\frac{f''(z)}{f'(z)}\sim-\frac{2}{z-p^*}$$

for $z$ near $p^*$. We skip the argument that shows that

$$\frac{f''(z)}{f'(z)}=\sum_{k=1}^{K}\beta_k\left(\frac{1}{z-z_k}-\frac{1}{z-p}-\frac{1}{z-p^*}\right)$$

in this case.

For the doubly connected case we consider first the map from the interior of an annulus to the exterior of two polygons; see Figure 1. We may take the annulus to be $\Omega=\{\mu<|z|<1\}, q\in\Omega$ with $f(q)=\infty$, vertices $w_{k,1}, 1\le k\le K_1$ (indexed counterclockwise) with turning angles $\beta_{k,1}\pi$ on one of the polygons, $w_{k,0}, 1\le k\le K_0$ (indexed clockwise) with turning angles $\beta_{k,0}\pi$ on the other, and $f$ mapping $|z|=\mu$ to the first polygon with $f(z_{k,1})=w_{k,1}$, $|z|=1$, to the other with $f(z_{k,0})=w_{k,0}$.

This mapping function $f$ can be constructed directly by reflection. We analytically continue $f$ across each arc of $|z|=\mu$ to obtain an analytic function on $\mu^2<|z|<\mu$. The function obtained is multivalued since different reflections are obtained across different arcs, but the global analytic function obtained has a single-valued pre-Schwarzian $f''(z)/f'(z)$. We do the same thing across arcs of

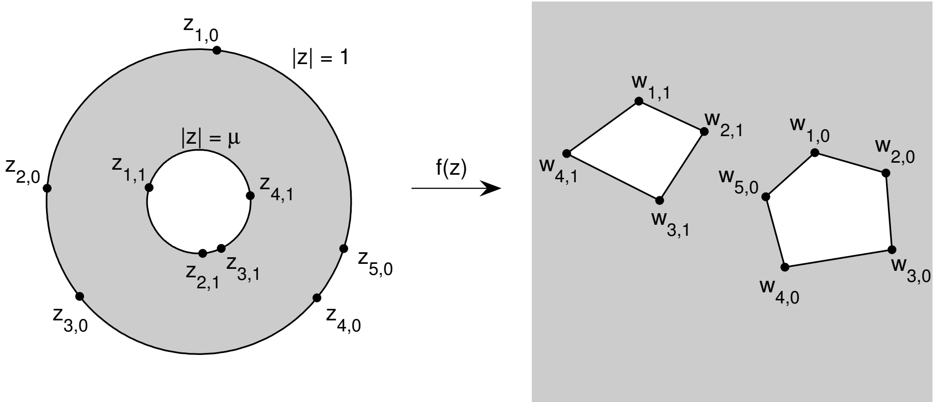


FIGURE 1. Notation for the map from an annulus to the exterior of two polygons.

$|z| = 1$ to $1 < |z| < 1/\mu$. Then the process can be continued both inwards and outwards to obtain a global analytic function on $\mathbb{C} \setminus V \setminus \{0\}$ where

$$V := \bigcup_{j=-\infty}^{\infty} \left( \bigcup_{k=1}^{K_0} \{\mu^{2j} z_{k,0}\} \cup \bigcup_{k=1}^{K_1} \{\mu^{2j} z_{k,1}\} \cup \{\mu^{2j} q\} \cup \{\mu^{2j} q^{-1}\} \right)$$

are the reflections of the prevertices and the pole $q$. (For more details see [4], where the bounded problem was dealt with.) The same local analysis as before shows that $f''(z)/f'(z) - \beta_{k,l}/(z - \mu^{2j} z_{k,l})$, $j = \dots, -1, 0, 1, \dots$, $l = 0, 1$ is analytic near $\mu^{2j} z_{k,l}$. This leads to the definition

$$\begin{aligned} S(z) = & \sum_{k=1}^{K_0} \left[ \beta_{k,0} \sum_{j=-\infty}^{\infty} \left( \frac{1}{z - z_{k,0}\mu^{2j}} - \frac{1}{z - q\mu^{2j}} \right) \right] \\ & + \sum_{k=1}^{K_1} \left[ \beta_{k,1} \sum_{j=-\infty}^{\infty} \left( \frac{1}{z - z_{k,1}\mu^{2j}} + \frac{1}{z - q^{-1}\mu^{2j}} \right) \right] \end{aligned}$$

of the singularity function for this problem. The rearrangement

$$\begin{aligned} S(z) = & \sum_{j=0}^{\infty} \sum_{k=1}^{K_0} \beta_{k,0} \left( \frac{1}{z - z_{k,0}\mu^{-2j}} - \frac{1}{z - q\mu^{-2j}} \right. \\ & \qquad \left. + \frac{z_{k,0}\mu^{2(j+1)}}{z(z - z_{k,0}\mu^{2(j+1)})} - \frac{q\mu^{2(j+1)}}{z(z - q\mu^{2(j+1)})} \right) \\ & + \sum_{j=0}^{\infty} \sum_{k=1}^{K_1} \beta_{k,1} \left( \frac{z_{k,1}\mu^{2j}}{z(z - z_{k,1}\mu^{2j})} - \frac{1}{z - q^{-1}\mu^{-2j}} \right. \\ & \qquad \left. + \frac{z_{k,1}\mu^{2(j+1)}}{z - z_{k,1}\mu^{-2(j+1)}} - \frac{q^{-1}\mu^{2(j+1)}}{z(z - q^{-1}\mu^{2(j+1)})} \right) \end{aligned}$$

follows from, e.g.

$$\frac{1}{z - z_{k,0}\mu^{2(j+1)}} = \frac{1}{z} + \frac{\frac{z_{k,0}\mu^{2(j+1)}}{z^2}}{1 - \frac{z_{k,0}\mu^{2(j+1)}}{z}},$$

$$\frac{1}{z - z_{k,1}\mu^{2j}} = \frac{1}{z} + \frac{\frac{z_{k,1}\mu^{2j}}{z^2}}{1 - \frac{z_{k,1}\mu^{2j}}{z}}$$

and $\sum_{k=1}^{K_0} \beta_{k,0} = -\sum_{k=1}^{K_1} \beta_{k,1} = 2$, (see [4]) and this form makes the convergence transparent. By construction $f''(z)/f'(z) - S(z)$ is analytic in $0 < |z| < \infty$. With this form of $S(z)$ we can readily show that $\mathrm{Re}(zS(z)) = -1$ for $|z| = \mu, 1$ between prevertices and deduce by continuity and the Maximum Principle that $\mathrm{Re}(zf''(z)/f'(z) - zS(z)) = 0$ for $\mu < |z| < 1$. The imaginary part is then constant. A simple argument [4], omitted here, shows the constant is zero. We have used an annulus here as a canonical domain to make the reflections easier to visualize, but replacing $\Omega$ with the exterior of two disks is easily done by using the explicit map $\zeta = (z-p)/(1-\overline{p}z)$ for $z$ outside the unit disk and outside another disjoint disk with $p$ inside this second disk. If $q = -1/\overline{p}$, the image of $\infty$, then $f((z-p)/(1-\overline{p}z))$ is the required map. We remark that the formula

$$f'(z) = A \prod_{k=1}^{K_0} \left[\Theta\left(\frac{z}{\mu z_{k,0}}\right)\right]^{\beta_{k,0}} \prod_{k=1}^{K_1} \left[\Theta\left(\frac{\mu z}{z_{k,1}}\right)\right]^{\beta_{k,1}} \left[\Theta\left(\frac{\zeta}{\mu q}\right)\Theta\left(\frac{q\zeta}{\mu}\right)\right]^{-2}$$

where

$$\Theta(w) = \prod_{j=0}^{\infty} \left(1 - \mu^{2j+1} w\right)\left(1 - \frac{\mu^{2j+1}}{w}\right)$$

can be derived from our formula, but we will make no use of this here.

**2.2. The general formula.** In [5], we have derived a general formula for the map from the exterior $\Omega$ of $m \geq 2$ disjoint disks to the exterior of $m$ polygons $P$. The idea is to find a singularity function $S(z)$ such that

$$\frac{f''(z)}{f'(z)} = S(z)$$

for $f$, the map from $f(z) = z + \mathcal{O}(1/z)$, $z \to \infty$. This is done by reflection as was reviewed in the simply connected cases discussed above. For higher connectivity there is a hierarchy of levels of reflection in which the number of reflections at each level increase by a factor of $m-1$, so that there is an exponential increase in the number of terms required in $S(z)$. At the first level $\Omega$ is reflected through each of the bounding circles to obtain bounded $m$ connected regions inside of each circle. At the next level each of these is reflected through each of its interior $m-1$ bounding circles. The process continues *ad infinitum.* The process is indicated in Figure 2 below for $m = 3$ and 3 levels.

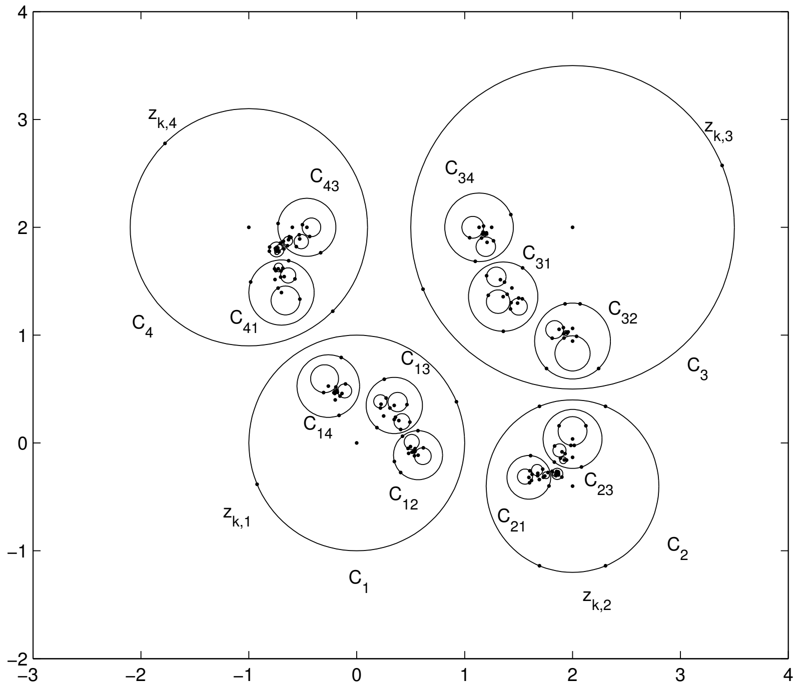


FIGURE 2. Reflections of $m = 4$ circles to level $N = 2$.

We have introduced the multi-index notation

$$\sigma_n = \{\nu_1\nu_2\cdots\nu_n : 1 \le \nu_j \le m,\, \nu_k \ne \nu_{k+1},\, k = 1, ..., n-1\}$$

for $n > 0$, $\sigma_0 = \phi$, in which case $\nu i = i$, and

$$\sigma_n(i) = \{\nu \in \sigma_n : \nu_n \ne i\}\,.$$

The Schwarz-Christoffel formula derived in [5] could then be written

$$f'(z) = A \prod_{i=1}^{m}\prod_{k=1}^{K_i}\prod_{\substack{j=0\\ \nu\in\sigma_j(i)}}^{\infty} \left(\frac{\zeta - z_{k,\nu i}}{\zeta - s_{\nu i}}\right)^{\beta_{k,i}} \tag{2}$$

where $\nu i = \nu_1\cdots\nu_n i$ if $\nu = \nu_1\cdots\nu_n$, $z_{k,\nu i}$ is the $n$-fold reflection of the prevertex $z_{k,i}$ successively through circles $C_{\nu_n}, \ldots, C_{\nu_2}, \ldots, C_{\nu_1}$. Similarly $s_{\nu i}$ is the $n$-fold reflection of the center $c_i$ of the circle $C_i$ bounding $\Omega$.

Now $f'(z)$ is obtained by integrating and exponentiating $S(z) = f''(z)/f'(z)$ where the singularity function $S(z)$ is given by

$$S(z) = \sum_{j=0}^{\infty}\sum_{i=1}^{m}\sum_{\nu\in\sigma_j(i)} \left(\sum_{k=1}^{K_i} \frac{\beta_{k,i}}{z - z_{k,\nu i}} - \frac{2}{z - s_{\nu i}}\right).$$

This result assumes convergence of the partial sums, $S_N(z)$, of the series above, truncated after $N$ levels of reflection $j = 0, \ldots, N$, to $S(z)$. Convergence has been proven when the disks in the circular domain are sufficiently well separated [5]. The rate of convergence is expressed in terms of an explicit separation parameter, $\Delta$, defined in Section 4, below.

The key idea in the proof that this formula is correct is to show that

$$\mathrm{Re}((z - c_i)S(z)) = -1 \tag{3}$$

on each of the circles $|z - c_i| = r_i$ bounding the circle domain $\Omega$. This is the $m$-connected analogue of the condition $\mathrm{Re}(zS(z)) = -1$ for simply and doubly connected problems.

We now have to formulate a set of equations for determining the free parameters in (2) to get a conformal mapping from the circle domain $\Omega$ to $P$.

The condition analogous to (1) is

$$\sum_{i=1}^{m}\sum_{k=1}^{K_i}\sum_{\substack{j=0\\ \nu\in\sigma_j(i)}}^{\infty} \beta_{k,i}\left(z_{k,\nu i} - s_{\nu i}\right) = 0. \tag{4}$$

We will not impose this condition as an equation; it will be used as a check on our computations.

## 3. Formulation of equations

From the general theory [8], [9] of multiply connected maps it follows that given $P$ and the normalization condition $f(z) = z + \mathcal{O}(1/z)$ for $z \approx \infty$, there is a unique circular domain $\Omega$ and a unique $f$ such that $f$ maps $\Omega$ onto $P$.

The parameters available to solve the problem are the prevertices $z_{k,i}$ such that $w_{k,i} = f(z_{k,i})$ for $k = 1, \ldots, K_i$, $i = 1, \ldots, m$, where $w_{k,i}$ are the corners of $\partial P = \Gamma = \Gamma_1 \cup \cdots \cup \Gamma_m$. These prevertices are parameterized as $z_{k,i} = c_i + r_i e^{i\theta_{k,i}}$, where

$$\theta_{1,i} < \theta_{2,i} < \cdots < \theta_{K_i,i}, \qquad i = 1, \ldots, m.$$

The $c_i$'s, $r_i$'s, and $\theta_{k,i}$'s are

$$K_1 + K_2 + \cdots + K_m + 3m$$

real parameters. They are also exactly the parameters needed to determine the Schwarz-Christoffel product

$$p(z) = \prod_{i=1}^{m}\prod_{k=1}^{K_i}\prod_{\substack{j=0\\ \nu\in\sigma_j(i)}}^{\infty} \left(\frac{z - z_{k,\nu i}}{z - s_{\nu i}}\right)^{\beta_{k,i}}.$$

The normalization $f(z) = z + \mathcal{O}(1/z)$ at infinity is difficult to impose on our formula, so we relax this to

$$f(z) = Az + B + \mathcal{O}\left(\frac{1}{z}\right) \tag{5}$$

and formulate equations which determine $A$ and $B$ implicitly. Our formulation of these equations borrows an idea from the original SCPACK of Trefethen [10]. We fix $c_1 = 0, r_1 = 1$, and $\theta_{1,1} = 0$. This removes four of the disposable real parameters; it can be done because we use (5). Then writing

$$f(z) = C \int_{z_{1,1}}^{z} p(z)\, dz + D \tag{6}$$

we can say $D = f(z_{1,1}) = w_{1,1}$ and

$$C = \frac{w_{2,1} - w_{1,1}}{\int_{z_{1,1}}^{z_{2,1}} p(z)\, dz}. \tag{7}$$

This leaves $K_1 + \cdots + K_m + 3m - 4$ real parameters. (We may think of $A$ and $B$ as being determined by $C$ and $D$.) There are $K_1 - 1$ side lengths to impose on the first polygon (the definition of $C$ automatically determines the first side length) and $K_2 + \cdots + K_m$ side lengths for the remaining polygons. These can be written as *side length conditions*:

$$|f(z_{k+1,i}) - f(z_{k,i})| = |w_{k+1,i} - w_{k,i}|, \qquad \begin{matrix} i = 1, \ldots, m, \\ k = 1, \ldots, K_i, \end{matrix} \ (k, i) \neq (1, 1).$$

There are $2(m - 1)$ (real) equations to fix *positions* of $\Gamma_1, \Gamma_2, \ldots, \Gamma_m$:

$$f(z_{1,i}) - f(z_{1,1}) = w_{1,i} - w_{1,1}, \qquad i = 2, \ldots, m.$$

and $m - 1$ equations to fix the *orientations* of $\Gamma_1, \Gamma_2, \ldots, \Gamma_m$:

$$\arg(f(z_{2,i}) - f(z_{1,i})) = \arg(w_{2,i} - w_{1,i}), \qquad i = 2, \ldots, m.$$

Note that these can be combined with the side length conditions for $k = 1$ as:

$$f(z_{2,i}) - f(z_{1,i}) = w_{2,i} - w_{1,i}, \qquad i = 2, \ldots, m.$$

We now have a total of $K_1 + K_2 + \cdots + K_m + 3m - 4$, (real) equations, exactly as many as we need.

Variations of this setup, such as integrating between neighboring circles to fix positions instead of always from $C_1$, can be substituted, as long as the number and independence of the conditions is preserved. The automatic selection of the best set of conditions is a topic for future research.

The equation

$$\frac{f''(z)}{f'(z)} = S(z) = \mathcal{O}\left(\frac{1}{z^3}\right)$$

at $z = \infty$, which is equivalent to (4), is not imposed explicitly. It is required for closure of the polygons, and we use it as a check on accuracy. Note that at least $S(z) = \mathcal{O}(1/z^2)$ is built in to our form of $S(z)$.

## 4. Examples

The evaluation of the integrals defining the $K_1 + \cdots + K_m + 3m - 4$ conditions derived in the previous section is done here using Gauss-Jacobi quadrature on the circular arcs and straight line paths naturally associated with the geometry. For the examples presented here, a relatively simple procedure sufficed: a fixed number of Gauss-Jacobi points are used for each integral (typically 30 points) and the possible crossing of straight line paths with another circle is ignored. In the next generation of the evolution of the computational algorithm, compound quadrature [10] rules will be introduced and a path generation procedure which avoids crossing a circle will be added. As the number of sides and components are increased this will almost certainly become necessary.

For a given set of $K_1 + \cdots + K_m + 3m - 4$ parameters, fixing $C$ and $D$ using (6) and (7) and then approximating the remaining integrals using quadrature yields the value of an objective function for the solution of the mapping problem. The solutions we present are obtained by using this function together with the continuation algorithm CONTUP of Allgower and Georg [1, Program 3].

In order to keep on the correct branch of the $\beta_{k,i}$, we rewrite the product formula as follows and truncate after $N$ levels of reflection:

$$p(z) = A \prod_{i=1}^{m} \prod_{k=1}^{K_i} \prod_{\substack{j=0 \\ \nu \in \sigma_j(i)}}^{\infty} \left( \frac{z - z_{k,\nu i}}{z - s_{\nu i}} \right)^{\beta_{k,i}} \approx A \prod_{i=1}^{m} \prod_{k=1}^{K_i} \prod_{\substack{j=0 \\ \nu \in \sigma_j(i)}}^{N} \left( 1 - \frac{z_{k,\nu i} - s_{\nu i}}{z - s_{\nu i}} \right)^{\beta_{k,i}}.$$

The truncations after $N$ levels of reflection are based on the method of images. Successive reflections can be represented using the following lemma, where $\rho_\nu(a)$ denotes reflection of $a$ through circle $C_\nu$.

**Lemma 1** (cf. [5, Lem. 1])**.** $\rho_{21}(\rho_2(a)) = \rho_2(\rho_1(a))$.

For example, $z_{k,1201} = \rho_1(\rho_2(\rho_0(z_{k,1}))$ and $C_{210} = \rho_{21}(C_{20}) = \rho_{21}(\rho_2(C_0)) = \rho_2(\rho_1(C_0))$. We have developed a MATLAB code based on these observations to produce the reflections of the prevertices and centers for use in our truncated product formula.

The convergence results derived in [5] are formulated in terms of a separation parameter $\Delta$ for the circular boundary components of the computational domain. Specifically, if $C_j$ and $C_k$ are two disjoint mutually exterior circles then they are non-overlapping if and only if $(r_j + r_k)/\left|c_j - c_k\right| < 1$ where $r_j$ is the radius and $c_j$ the center of a circle. The quantity $\Delta$ is the largest of these separation quotients between all pairs of boundary circles. Our convergence estimates in [5] show that convergence is rapid if the circles are well-separated, i.e. $\Delta$ is small. We have preliminary theoretical estimates on numerical errors based on our theory in [5] and depending on the parameter $\Delta$.

To give estimates for our numerical errors, we will use the following theorem from [5] which estimates, for general $m$, how accurately $S_N(z)$, the series for

$S(z)$ truncated after $N$ levels of reflection, satisfies the boundary conditions (3) for $S(z) = f''(z)/f'(z)$.

**Theorem 1.** *If* $\Delta < (m-1)^{-1/4}$ *then for* $z \in C_i$, $z \neq z_{k,i}$

$$\mathrm{Re}((z - s_i)S_N(z)) = -1 + \mathcal{O}((\Delta^2\sqrt{m-1})^N)$$

*and* $\mathrm{Re}((z - s_i)S(z)) = -1$.

In [5], we note that this convergence rate is not optimal for the known case $m = 2$. This suggests the convergence results can be improved for the general case, $m \geq 2$. Indeed, we have observed convergence numerically when the condition of Theorem 1 is violated.

In our first example, $f$ maps to the exterior of two collinear slits $[-1, 1]$ and $[2, 5]$; Figure 3. Table 1 gives results to illustrate the decrease in the truncation error with respect to the decreasing radii $r_\nu$ of the reflected circles as $N$ increases. Bounds on the rate at which the sums of $r_\nu$ and the areas $\pi {r_\nu}^2$ decrease as

| $N$ | $\Delta^{4N} \sum_{i=1}^{3} {r_i}^2$ | $\sum_{\|\nu\|=N+1} {r_\nu}^2$ | $\sum_{\|\nu\|=N+1} r_\nu$ | max bc error |
|---|---|---|---|---|
| 0 | $1.04 \cdot 10^{0}$ | $1.04 \cdot 10^{0}$ | $1.42 \cdot 10^{0}$ | $3.54 \cdot 10^{-2}$ |
| 1 | $2.77 \cdot 10^{-2}$ | $1.74 \cdot 10^{-3}$ | $5.82 \cdot 10^{-2}$ | $1.54 \cdot 10^{-3}$ |
| 2 | $7.41 \cdot 10^{-4}$ | $3.25 \cdot 10^{-6}$ | $2.52 \cdot 10^{-3}$ | $5.67 \cdot 10^{-5}$ |
| 3 | $1.98 \cdot 10^{-5}$ | $6.07 \cdot 10^{-9}$ | $1.09 \cdot 10^{-4}$ | $2.85 \cdot 10^{-6}$ |
| 4 | $5.30 \cdot 10^{-7}$ | $1.13 \cdot 10^{-11}$ | $4.70 \cdot 10^{-6}$ | $1.06 \cdot 10^{-7}$ |
| 5 | $1.42 \cdot 10^{-8}$ | $2.12 \cdot 10^{-14}$ | $2.03 \cdot 10^{-7}$ | $5.32 \cdot 10^{-9}$ |
| 6 | $3.79 \cdot 10^{-10}$ | $3.96 \cdot 10^{-17}$ | $8.77 \cdot 10^{-9}$ | $1.98 \cdot 10^{-10}$ |
| 7 | $1.01 \cdot 10^{-11}$ | $7.39 \cdot 10^{-20}$ | $3.79 \cdot 10^{-10}$ | $9.78 \cdot 10^{-12}$ |
| 8 | $2.71 \cdot 10^{-13}$ | $1.38 \cdot 10^{-22}$ | $1.64 \cdot 10^{-11}$ | $8.97 \cdot 10^{-13}$ |

Table 1. Decrease of radii of reflected circles and error in boundary conditions for the example of two collinear slits in Figure 3.

the number of circles increase with each level of reflection are used to estimate the convergence rate and how well the boundary conditions (max bc error) are satisfied. A version of our non-linear equations that fixes the centers $c_1$ and $c_2$ is used in this example. The values of the $\theta_{k,i}$'s are solved for here but remain at about 0 and $2\pi$ with nearby initial guesses. The initial guesses for the radii are both 1, and the final values are $r_1 = .57688$ and $r_2 = .83848$ giving $\Delta = .40439$. The homotopy search method [1] quickly finds a good initial guess for the Newton method.

In Figures 3 and 4, the dots on a Cartesian grid in the circle plane are mapped to the dots in the slit planes using a truncated Laurent series approximation to $f(z)$

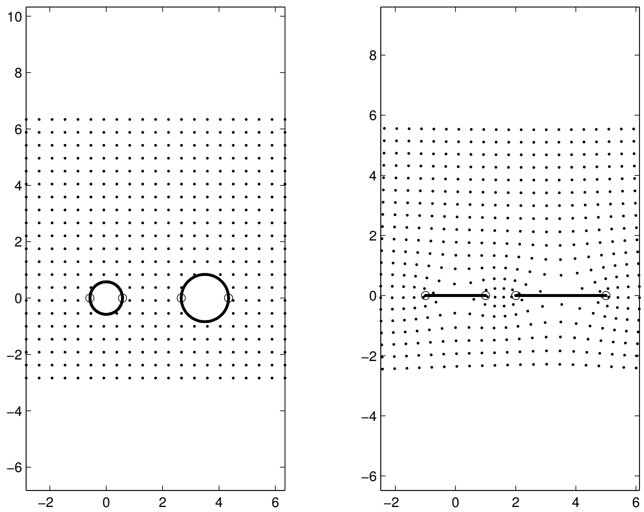


Figure 3. Map to the exterior of two collinear slits for Table 1.

as in [6, eq. (3)]. Since the normalization of $f$ at infinity is $f(z) \approx Az + B$, the images of the Cartesian grids may be rotated or distorted when $A \neq 1$ and $B \neq 0$.

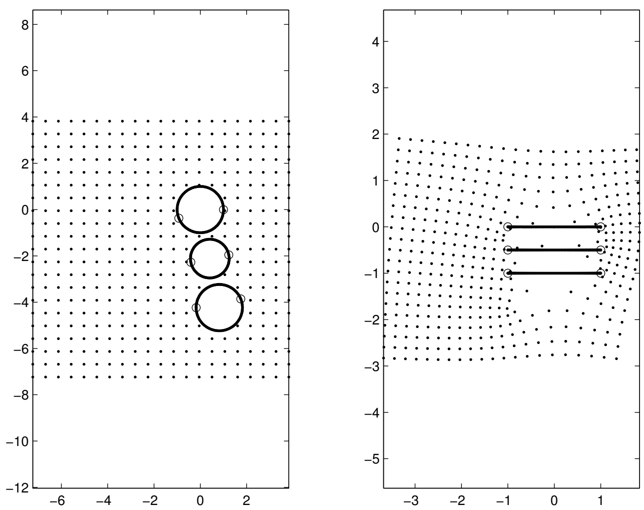


Figure 4. Crowding of circles due to elongated channels between the slits.

Figure 4 shows an example of a phenomenon which may be called a type of "crowding" for the multiply connected case. Here elongated channels between the slits cause the conformally equivalent circles to crowd together. This is somewhat similar to a more (exponentially) severe crowding of the prevertices for the simply connected Schwarz-Christoffel map from the disk to an elongated rectangle [7]. Note that the nearness of the slit tips in Figure 3 does not cause a similar crowding of the circles.

The examples in Figures 5 and 6 were motivated by the work of van Deursen [11, 12] on electromagnetic isolation of circuit elements. We see the equipotential and force lines for the cross section of a cabinet and circuit element in Figure 5. Van Deursen used the doubly connected Schwarz-Christoffel formula. In Figure 6,

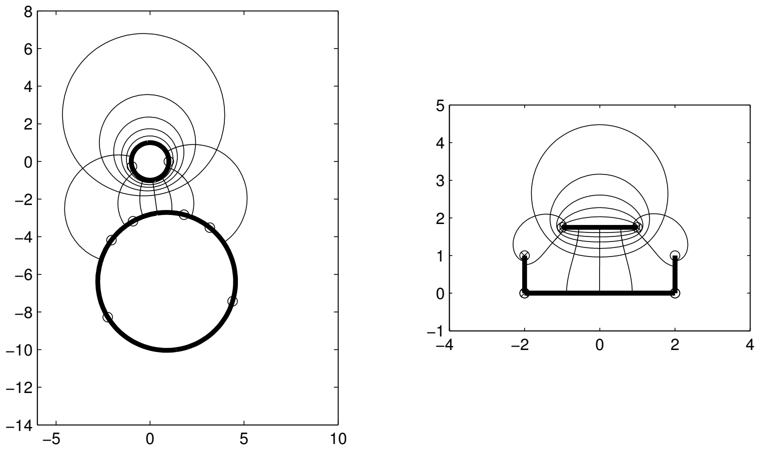


FIGURE 5. Example motivated by van Deursen paper.

a two part circuit element illustrates the potential application of our formulae for higher connectivity to these circuit design problems.

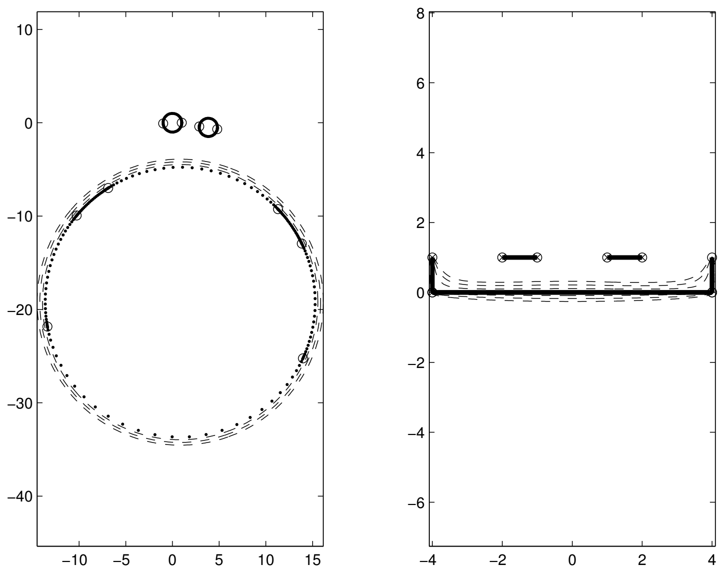


FIGURE 6. Map to circuit element with $m = 3$.

As in Figures 7 and 8, the images of concentric circles around the circles in $\Omega$ in Figure 6 are plotted. The values of $w$ are computed using Laurent series [6], whereas the level curves in Figure 5 are computed using quadrature to find the images of the level curves in $\Omega$; these are given by an explicit map from an annulus.

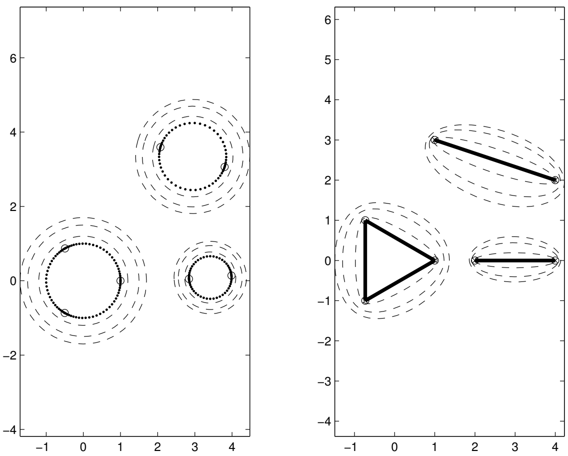

FIGURE 7. Map for $m = 3$ to be compared with Figure 8.

Our last example, shown in Figures 7 and 8, shows maps to regions of higher connectivity. Increasing the connectivity from $m = 3$ in Figure 7 to $m = 4$ in Figure 8 increased the computing time dramatically, from 34 to 413 CPU seconds on a PC (a factor of about 12). This shows a need for introducing more efficient evaluation of the products in (2), an issue that is under study.

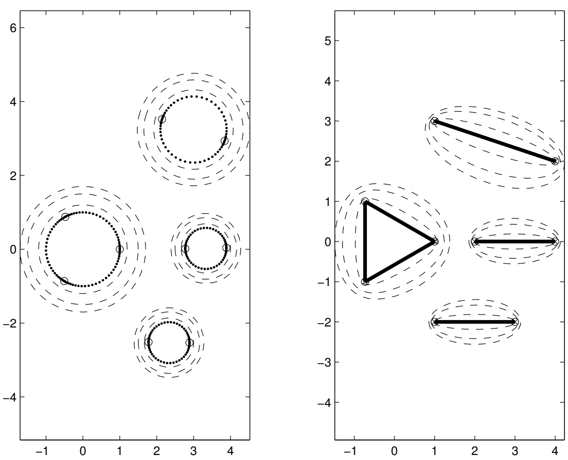

FIGURE 8. Map for $m = 4$ case.

## References


1. E. L. Allgower and K. Georg, *Numerical Continuation Methods: An Introduction*, Springer-Verlag, New York 1990.
2. D. Crowdy, The Schwarz-Christoffel mapping to bounded multiply connected polygonal domains, *Proc. Royal Society A* **461** (2005), 2653–2678.
3. T. K. DeLillo, Schwarz-Christoffel mapping of bounded, multiply connected domains, *Comput. Methods Funct. Theory* **6** no.2 (2006), 275–300.
4. T. K. DeLillo, A. R. Elcrat, and J. A. Pfaltzgraff, Schwarz-Christoffel mapping of the annulus, *SIAM Review* **43** no.3 (2001), 469–477.
5. ———, Schwarz-Christoffel mapping of multiply connected domains, *J. d'Analyse Math.* **94** (2004), 17–47.
6. T. K. DeLillo, M. A. Horn, and J. A. Pfaltzgraff, Numerical conformal mapping of multiply connected regions by Fornberg-like methods, *Numer. Math.* **83** (1999), 205–230.
7. T. A. Driscoll and L. N. Trefethen, *Schwarz-Christoffel Mapping*, Cambridge University Press, Cambridge 2002.
8. G. M. Golusin, *Geometric Theory of Functions of a Complex Variable*, Amer. Math. Soc., Rhode Island 1969.
9. P. Henrici, *Applied and Computational Complex Analysis, vol.3*, John Wiley and Sons, New York 1986.
10. L. N. Trefethen, Numerical computation of the Schwarz-Christoffel transformation, *SIAM J. Sci. Stat. Comput.* **1** (1980), 82–102.
11. A. P. J. van Deursen, Schwarz-Christoffel analysis of cable conduits with noncontacting cover, *IEEE Electronic Letters*, **37** no.13 (2001), 816–817.
12. A. P. J. van Deursen and S. Kapora, Reduction of common mode coupling of printed circuits boards by nearby U-shaped metal panel, to appear in *IEEE Trans. on Electromagnetic Compatibility.*



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